An equation involving the Fibonacci numbers and Smarandache primitive function

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***Abstract*—for any positive integer n, let $S_p(n)$ denotes the Smarandache primitive function, F_n denotes the Fibonacci numbers. The main purpose of this paper is using the elementary methods to study the number of the solutions of the equation $S_p(F_1)+S_p(F_2)+\ldots+S_p(F_n)=S_p(F_{n+2}-1)$, and give all positive integer solutions for this equation.**

Keywords- Fibonacci numbers; Smarandache primitive function; Equation; Solutions

I. INTRODUCTION

As usual, the Fibonacci sequence F_n is defined by the second-order linear recurrence sequence $F_{n+2}=F_{n+1}+F_n$ for $n\geq 1$, $F_1=1$, $F_2=1$. This sequence plays a very important role in the studied of theory and application of mathematics. Therefore, the various properties of F_n was investigated by many authors. For example, R.L. Duncan [1] and L. Kuipers [2] proved that ($\log F_n$) is uniformly distributed mod I. Fengzhen, Zhao [3] and Kh. Bibak [4] obtained some identities involving the Fibonacci numbers.

Let p be a prime, n be any positive integer. The Smarandache primitive function $S_p(n)$ is defined as the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_2(1)=2$, $S_2(2)=S_2(3)=4$, $S_2(4)=6$, $\cdots$.

In problem 47, 48 and 49 of book [5], the famous Rumanian born American number theorist, Professor F.Smarandache asked us to study the properties of the $S_p(n)$. There are closely relations between the Smarandache primitive function $S_p(n)$ and the famous function $S(n)$, where

$$S(n)=\min\{m: m\in N, n|m!\}.$$

From the definition of $S(n)$, obviously we have $S(p)=p$, and if $n\neq 4$, $n\neq p$, then $S(n)<n$. So we have

$$\pi(x)=-1+\sum_{i=2}^{[x]}\left[\frac{S(n)}{n}\right],$$

where $\pi(x)$ denotes the number of primes which less than x.

The research on Smarandache function $S(n)$, Smarandache primitive function $S_p(n)$ and the equations involving Smarandache primitive function $S_p(n)$ is an significant and important problem in Number Theory. Therefore, many scholars and researchers have studied them before, see reference [6-8]. Professor Zhang Wenpeng [9] has obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n, we have

$$S_p(n)=(p-1)n+O\left(\frac{p}{\ln p}\cdot\ln n\right).$$

It seems that no one knows the relationship between the Fibonacci numbers and the Smarandache primitive function. In this paper, we use the elementary methods to study the solvability of the equation

$$S_p(F_1)+S_p(F_2)+\cdots+S_p(F_n)=S_p(F_{n+2}-1),$$

and give all positive integer solutions for this equation. That is, we will prove the following:

Theorem Let p be a given prime, n be any positive integer, then the equation

$$S_p(F_1)+S_p(F_2)+\cdots+S_p(F_n)=S_p(F_{n+2}-1), \qquad (1)$$

has finite solutions. They are $n=1,2,\cdots,n_p$, where

$$n_p=\left[\frac{\log\left(\sqrt{5}(p+1)+\sqrt{5(p+1)^2+4}\right)-\log 2}{\log\left(1+\sqrt{5}\right)-\log 2}-2\right],$$

$[x]$ denotes the biggest integer $\leq x$.

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Especially, taking $p=3,5,7$, we may immediately deduce the following:

Corollary 1 The positive integer solutions of the equation

$$S_3(F_1)+S_3(F_2)+\cdots+S_3(F_n)=S_3(F_{n+2}-1)$$

are $n=1,2$.

Corollary 2 The positive integer solutions of the equation

$$S_5(F_1)+S_5(F_2)+\cdots+S_5(F_n)=S_5(F_{n+2}-1)$$

are $n=1,2,3$.

Corollary 3 The positive integer solutions of the equation

$$S_7(F_1)+S_7(F_2)+\cdots+S_7(F_n)=S_7(F_{n+2}-1)$$

are $n=1,2,3,4$.

II. PRELIMINARIES

To complete the proof of the Theorem, we need the following several simple.

Lemma 1 Let p be a prime, n be any positive integer. $S_p(n)$ denotes the Smarandache primitive function, then we have

$$S_p(k)\begin{cases}=pk,\text{if } k\le p,\\ <pk,\text{if } k>p.\end{cases}$$

Proof (See reference [10]).

Lemma 2 Let F_n be the Fibonacci sequence with $F_1=1$ and $F_2=1$, then we have the identity

$$F_1+F_2+\cdots+F_n=F_{n+2}-1.$$

Proof From the second-order linear recurrence sequence $F_{n+2}=F_{n+1}+F_n$ we can easily get the identity of Lemma 2.

Lemma 3 Let p be a prime, n be any positive integer, if n and p satisfying $p^{\alpha}\parallel n!$, then

$$\alpha=\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right].$$

Proof (See reference [11]).

Lemma 4 Let p be a prime, n be any positive integer. Then there must exist a positive integer m_k with $1\le m_k\le F_k\ (k=1,2,\cdots,n)$ such that

$$S_p(F_1)=m_1p, S_p(F_2)=m_2p,\cdots S_p(F_n)=m_np,$$

and

$$F_k\le\sum_{i=1}^{\infty}\left[\frac{m_kp}{p^i}\right].$$

Proof From the definition of $S_p(n)$, Lemma 1 and Lemma 3, we can easily get the conclusions of Lemma 4.

III. PROOF OF THE THEOREM

In this section, we will complete the proof of the Theorem.

We will discuss the solutions of (1) in the following four cases:

1) If $F_{n+2}-1\le p$, solving this inequality we can get $1\le n\le n_p$, where

$$n_p=\left[\frac{\log\left(\sqrt{5}(p+1)+\sqrt{5(p+1)^2+4}\right)-\log 2}{\log\left(1+\sqrt{5}\right)-\log 2}-2\right],$$

$[x]$ denotes the biggest integer $\le x$.

Then from Lemma 1 we have

$$S_p(F_{n+2}-1)=p(F_{n+2}-1).$$

Noting that $1\le n\le n_p$, that is $F_n\le F_{n+2}\le p$, from Lemma 1 and Lemma 2 we can get

$$S_p(F_1)+S_p(F_2)+\cdots+S_p(F_n)=p(F_{n+2}-1).$$

Combining above two formulae, we may easily get $n=1,2,\cdots,n_p$ are the solutions of (1).

2) If $F_n\le p\le F_{n+2}-1$, solving this inequality we can get $n_p\le n\le n'_p$, where

$$n_p=\left[\frac{\log\left(\sqrt{5}(p+1)+\sqrt{5(p+1)^2+4}\right)-\log 2}{\log\left(1+\sqrt{5}\right)-\log 2}-2\right]$$

and

$$n'_p=\left[\frac{\log\left(\sqrt{5}p+\sqrt{5p^2+4}\right)-\log 2}{\log\left(1+\sqrt{5}\right)-\log 2}\right],$$

$[x]$ denotes the biggest integer $\le x$.

Then from Lemma 1 we have

$$S_p(F_{n+2}-1)\le p(F_{n+2}-1).$$

But

$$S_p(F_1)+S_p(F_2)+\cdots+S_p(F_n)=p(F_{n+2}-1).$$

Hence (1) has no solution in this case.

3) If $n=n'_p+1$, that is $p<F_n<F_{n+2}-1$.

a) If $F_n=F_{n'_p+1}=p+1$, then $m_{n'_p+1}=p$.

Notice that $m_1 = F_1, m_2 = F_2, \cdots, m_{n'_p} = F_{n'_p} \le p$. So we have

$$S_p(F_1) + S_p(F_2) + \cdots + S_p(F_n) = p(m_1 + m_2 + \cdots + m_{n'_p+1})$$
$$= p(F_{n'_p+2} - 1) + p^2.$$

If $p = 2$, that is $n = n'_p + 1 = 4$. Then

$$S_2(F_1) + S_2(F_2) + S_2(F_3) + S_2(F_4)$$
$$= 2(F_5 - 1) + 2^2 = 12.$$

However, $S_2(F_{n+2} - 1) = S_2(F_6 - 1) = S_2(7) = 8$.

So there is no solution for (1) in this case.

If $p > 2$, notice that

$$\sum_{i=1}^{\infty} \left[\frac{p(F_{n'_p+2} - 1) + p^2 - 1}{p^i} \right]$$
$$= \left[\frac{p(F_{n'_p+2} - 1) + p(p-1) + (p-1)}{p} \right] + \left[\frac{p(F_{n'_p+1} + F_{n'_p} - 1) + p^2 - 1}{p^2} \right] + \sum_{i=3}^{\infty} \left[\frac{p(F_{n'_p+2} - 1) + p^2 - 1}{p^i} \right]$$
$$= (F_{n'_p+2} - 1) + (p-1) + \left[\frac{2p^2 + (pF_{n'_p} - 1)}{p^2} \right]$$
$$+ \sum_{i=3}^{\infty} \left[\frac{p(F_{n'_p+2} - 1) + p^2 - 1}{p^i} \right]$$
$$\ge (F_{n'_p+2} - 1) + (p-1) + 2$$
$$= F_{n'_p+2} + F_{n'_p+1} - 1$$
$$= F_{n'_p+3} - 1.$$

Then from Lemma 3 we can get

$$p^{F_{n'_p+3}-1} \,\Big|\, \left(p(F_{n'_p+2} - 1) + p^2 - 1\right)!.$$

Therefore, $S_p(F_{n+2} - 1) = S_p(F_{n'_p+3} - 1)$

$$\le p(F_{n'_p+2} - 1) + p^2 - 1$$
$$< p(F_{n'_p+2} - 1) + p^2$$
$$= S_p(F_1) + S_p(F_2) + \cdots + S_p(F_n).$$

So there is no solution for (1) in this case.

b) If $F_n = F_{n'_p+1} > p + 1$, then we have $p \ge 3$, otherwise, $F_n = F_{n'_p+1} = F_4 = p + 1$, moreover, we have $p < m_{n'_p+1} \le F_{n'_p+1} = F_{n'_p-1} + F_{n'_p} < 2p$.

If $p = 3$, that is $n = n'_p + 1 = 5$, then

$$S_3(F_1) + S_3(F_2) + S_3(F_3) + S_3(F_4) + S_3(F_5) = 33.$$

However, $S_3(F_{n+2} - 1) = S_3(F_7 - 1) = S_3(12) = 27$.

So there is no solution for (1) in this case.

If $p > 3$, from Lemma 4 we know there must exist a positive integer m_k with $1 \le m_k \le F_k \ (k = 1, 2, \cdots, n'_p + 1)$ such that

$$S_p(F_1) = m_1 p, S_p(F_2) = m_2 p, \cdots S_p(F_n) = S_p(F_{n'_p+1}) = m_{n'_p+1} p.$$

Then we have

$$S_p(F_1) + S_p(F_2) + \cdots + S_p(F_n) = p(m_1 + m_2 + \cdots + m_{n'_p+1}).$$

On the other hand, notice that $m_1 = F_1, m_2 = F_2, \cdots, m_{n'_p} = F_{n'_p}$ and $p < m_{n'_p+1} < 2p$.

Then from Lemma 4 we have

$$\sum_{i=1}^{\infty} \left[\frac{(m_1 + m_2 + \cdots + m_n)p - 1}{p^i} \right]$$
$$= \sum_{i=1}^{\infty} \left[\frac{p(m_1 + m_2 + \cdots + m_{n'_p+1} - 1) + p - 1}{p^i} \right]$$
$$\ge m_1 + m_2 + \cdots + m_{n'_p+1} + \left(m_{n'_p+1} + \sum_{i=2}^{\infty} \left[\frac{m_{n'_p+1}}{p^i} \right] \right)$$
$$\ge \sum_{i=2}^{\infty} \left[\frac{m_1 p}{p^i} \right] + \sum_{i=2}^{\infty} \left[\frac{m_2 p}{p^i} \right] + \cdots + \sum_{i=2}^{\infty} \left[\frac{m_{n'_p+1} p}{p^i} \right]$$
$$\ge F_1 + F_2 + \cdots + F_{n'_p+1}$$
$$= F_{n'_p+3} - 1.$$

Then from Lemma 3 we can get

$$p^{F_{n'_p+3}-1} \,\Big|\, \left(p(m_1 + m_2 + \cdots + m_{n'_p+1}) - 1\right)!.$$

Therefore,

$$S_p\left(F_{n+2}-1\right)=S_p\left(F_{n'_p+3}-1\right)$$
$$\le p\left(m_1+m_2+\cdots+m_{n'_p+1}\right)-1$$
$$< p\left(m_1+m_2+\cdots+m_{n'_p+1}\right)$$
$$=S_p\left(F_1\right)+S_p\left(F_2\right)+\cdots+S_p\left(F_n\right).$$

Hence $n=n'_p+1$ is not a solution for (1).

4) If $n\ge n'_p+2$, that is $p<F_n<F_{n+2}-1$. So from Lemma 4 we know there must exist a positive integer m_k with $1\le m_k\le F_k\left(k=1,2,\cdots,n\right)$ such that

$$S_p\left(F_1\right)=m_1p, S_p\left(F_2\right)=m_2p,\cdots S_p\left(F_n\right)=m_np.$$

Then we have

$$S_p\left(F_1\right)+S_p\left(F_2\right)+\cdots+S_p\left(F_n\right)=p\left(m_1+m_2+\cdots+m_n\right).$$

On the other hand, if $p\ge 3$, from the analysis of (iii) we know there must exist a positive integer m_j with $p<m_j<2p\left(j=n'_p+1,n'_p+2,\cdots,n\right)$, then notice that

$m_1=F_1, m_2=F_2,\cdots,m_{n'_p}=F_{n'_p}$, from Lemma 4 we have

$$\sum_{i=1}^{\infty}\left[\frac{\left(m_1+m_2+\cdots+m_n\right)p-1}{p^i}\right]$$
$$=\sum_{i=1}^{\infty}\left[\frac{p\left(m_1+m_2+\cdots+m_n-1\right)+p-1}{p^i}\right]$$
$$\ge m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{p\left(m_1+m_2+\cdots+m_{n'_p+1}\right)+p-1}{p^i}\right]+\sum_{i=1}^{\infty}\left[\frac{p\left(m_{n'_p+1}+m_{n'_p+2}+m_n-1\right)}{p^i}\right]$$
$$=m_1+m_2+\cdots+m_{n'_p+1}-1+\sum_{i=2}^{\infty}\left[\frac{p\left(F_{n'_p+2}-1\right)+p-1}{p^i}\right]+\sum_{i=2}^{\infty}\left[\frac{m_{n'_p+1}+m_{n'_p+2}+m_n-1}{p^i}\right]$$
$$\ge m_1+m_2+\cdots+m_{n'_p}+\left(m_{n'_p+1}+\sum_{i=1}^{\infty}\left[\frac{m_{n'_p+1}}{p^i}\right]\right)+\left(m_{n'_p+2}+\sum_{i=1}^{\infty}\left[\frac{m_{n'_p+2}}{p^i}\right]\right)$$
$$+\cdots+\left(m_j+\sum_{i=1}^{\infty}\left[\frac{m_j-1}{p^i}\right]\right)+\cdots+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{p^i}\right]\right)$$
$$\ge\sum_{i=2}^{\infty}\left[\frac{m_1p}{p^i}\right]+\sum_{i=2}^{\infty}\left[\frac{m_2p}{p^i}\right]+\cdots+\sum_{i=2}^{\infty}\left[\frac{m_np}{p^i}\right]$$
$$\ge F_1+F_2+\cdots+F_n$$
$$=F_{n+2}-1.$$

Then from Lemma 3 we can get

$$p^{F_{n+2}-1}\left|\left(p\left(m_1+m_2+\cdots+m_n\right)-1\right)!\right..$$

Therefore,

$$S_p\left(F_{n+2}-1\right)\le p\left(m_1+m_2+\cdots+m_n\right)-1$$
$$<p\left(m_1+m_2+\cdots+m_n\right)$$
$$=S_p\left(F_1\right)+S_p\left(F_2\right)+\cdots+S_p\left(F_n\right).$$

If $p=2$, like the analysis above, it will be easy to get

$$S_2\left(F_1\right)+S_2\left(F_2\right)+\cdots+S_2\left(F_n\right)>S_2\left(F_{n+2}-1\right).$$

Hence $n\ge n'_p+2$ (1) has no solution.

Now the Theorem follows from *1)*, *2)*, *3)* and *4)*.

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